9-Apr L18
Wednesday, April 9, 2025 1:42 PM

§ 5.3 cont'd

Transposing a Vector

ex. given $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ then $\vec{v}^T = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

transpose allows us to express dot product of 2 columns as matrix multiplication (!):

thm: given $\vec{v}, \vec{w}$ are column vectors in $\mathbb{R}^n$
then $\vec{v} \cdot \vec{w} = \vec{v}^T \vec{w}$

ex. given $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

then $\vec{v} \cdot \vec{w} = \vec{v}^T \vec{w}$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}}_{1\times 3} \underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}_{3\times 1}$$ result: 1×1 (scalar)

$1(1) - 2(1) + 3(1)$

$2 = 2$ □

follow-up thm: given $n\times n$ matrix A
A is orthogonal iff $A^T A = I_n$
or
$A^{-1} = A^T$

Equivalent Statements

given $n \times n$ matrix A:

i. A is orthogonal
ii. $L(\vec{x}) = A\vec{x}$ preserves length or $\|A\vec{x}\| = \|\vec{x}\|$ $\forall \vec{x} \in \mathbb{R}^n$
transformation
iii. columns of A form an orthonormal basis of $\mathbb{R}^n$
iv. $A^T A = I_n$
v. $A^{-1} = A^T$
vi. A preserves dot product i.e., $(A\vec{x}) \cdot (A\vec{y}) = \vec{x} \cdot \vec{y}$ $\forall \vec{x}, \vec{y} \in \mathbb{R}^n$

Transpose Properties

vi. A preserves dot product i.e., $(A\vec{x})\cdot(A\vec{y}) = \vec{x}\cdot\vec{y}$

Transpose Properties

i. $(A+B)^T = A^T + B^T$ $m\times n$ A, B

ii. $(kA)^T = k\,(A^T)$ $m\times n$ A, k is scalar

iii. $(AB)^T = B^T A^T$ careful of order A is $m\times p$, B is $p\times n$

iv. $\text{rank}(A^T) = \text{rank}(A)$ $\forall A$

v. $(A^T)^{-1} = (A^{-1})^T \longleftarrow \forall$ invertible $n\times n$ A

Matrix of Orthogonal Projection

thm: given subspace V of $\mathbb{R}^n$ with orthonormal basis $\hat{u}_1, \ldots, \hat{u}_m$

then matrix P of orthogonal projection onto V is:

$$P = QQ^T \quad \text{where } Q = \begin{bmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_m \\ | & & | \end{bmatrix}$$

ex. determine matrix P of orthogonal projection onto subspace of $\mathbb{R}^4$ spanned by $\vec{u}_1 = \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$, $\vec{u}_2 = \frac{1}{2}\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}$

soln:

can easily show $\vec{u}_1, \vec{u}_2$ are orthonormal

$$\therefore P = QQ^T = \frac{1}{2}\cdot\frac{1}{2}\underbrace{\begin{bmatrix}1&1\\1&-1\\1&-1\\1&1\end{bmatrix}}_{4\times 2}\underbrace{\begin{bmatrix}1&1&1&1\\1&-1&-1&1\end{bmatrix}}_{2\times 4}$$

expect result: 4×4

work for 1^{ST} column:

$$\underbrace{\begin{bmatrix}1&1\\1&-1\\1&-1\\1&1\end{bmatrix}}_{4\times 2}\underbrace{\begin{bmatrix}1\\1\end{bmatrix}}_{2\times 1} = \underbrace{\begin{bmatrix}1+1\\1-1\\1-1\\1+1\end{bmatrix}}_{4\times 1} = \begin{bmatrix}2\\0\\0\\2\end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix}2&0&0&2\\0&2&2&0\\0&2&2&0\\2&0&0&2\end{bmatrix} \longleftarrow \text{divide all entries by 2}$$

(factor out 2, in other words)

$$P = \frac{1}{2}\begin{bmatrix}1&0&0&1\\0&1&1&0\\0&1&1&0\\1&0&0&1\end{bmatrix}$$

Inner Product Spaces

so far, studied concepts related to dot product
- vector length
- orthogonality

so far, studied concepts related to dot product
- vector length
- orthogonality

next, generalize dot product to inner product
use when linear space is not $\mathbb{R}^n$:

def'n: inner product in linear space V assigns a scalar to any f, g of elements of V

notation: $\langle f, g \rangle$

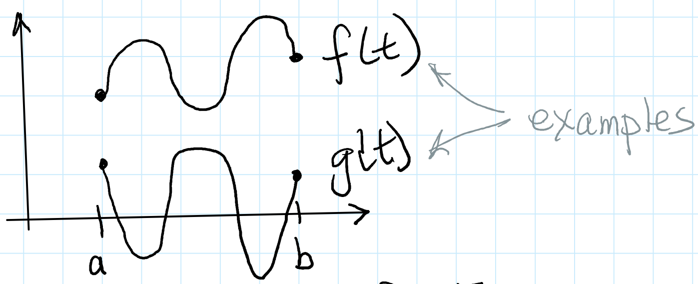
given $f, g, h \in V$ and $\forall c \in \mathbb{R}$, following properties hold:

- symmetry $\langle f, g \rangle = \langle g, f \rangle$
- distribution $\langle f+h, g \rangle = \langle f, g \rangle + \langle h, g \rangle$
- scalars: $\langle cf, g \rangle = c \langle f, g \rangle$
- positive definiteness: $\langle f, f \rangle = 0 \quad \forall$ non-zero $f \in V$

def'n: linear space endowed with an inner product is an inner product space

ex. given linear space $C[a,b]$, defined as all continuous functions whose domain is $[a,b]$ s.t. $a < b$

closed interval $\Rightarrow$ cont.



functions f and g in $C[a,b]$ $:= \langle f, g \rangle = \int_a^b f(t)g(t)\,dt$

proof of symmetry: $\langle f, g \rangle = \int_a^b f(t)g(t)\,dt$
$$= \int_a^b g(t)f(t)\,dt = \langle g, f \rangle \quad \square$$

ex. let l_2 = space of all "square summable" infinite sequences

i.e., $\vec{x} = (x_0, x_1, x_2, \dots, x_n, \dots)$
$$= \sum_{i=0}^{\infty} (x_i)^2 = (x_0)^2 + (x_1)^2 + (x_2)^2 + \dots \quad \text{converges}$$

$\sum_{i=0} (x_i) = (x_0) + (x_1) + (x_2) + \dots$

then can define inner product as $\langle x,y \rangle = \sum_{i=0}^{\infty} x_i y_i = x_0 y_0 + x_1 y_1 + \dots$

def'n: <u>trace</u> of a square matrix is the sum of diagonal entries

ex. $\operatorname{trace}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 1+4 = \boxed{5}$

in $\mathbb{R}^{n \times m}$ (space of all $n \times m$ matrices)

has an inner product $\langle A, B \rangle = \operatorname{trace}(A^T B)$

norm of element f of an inner product space: $\|f\| = \sqrt{\langle f,f \rangle}$

Likewise, given elements f and g are orthogonal if $\langle f,g \rangle = 0$

ex. in inner product space $C[0,1]$ with $\langle f,g \rangle = \int_0^1 fg\,dt$

find $\|f\|$ for $f(t) = t^2$

sol'n: $\|f\| = \sqrt{\langle f,f \rangle}$

$= \sqrt{\int_0^1 t^2 t^2\,dt}$

$= \sqrt{\int_0^1 t^4\,dt}$

$= \sqrt{\frac{t^5}{5}\Big|_0^1} = \sqrt{\frac{1}{5}(1-0)} = \sqrt{\frac{1}{5}}$ or $\boxed{\frac{1}{\sqrt{5}}}$

ex given $f(t) = \sin t$, $g(t) = \cos t$

show f and g are orthogonal in inner product space $C[0,2\pi]$

with $\langle f,g \rangle = \int_0^{2\pi} fg\,dt$

sol'n: $\langle f,g \rangle = \int_0^{2\pi} \sin t \underline{\cos t\,dt}$ $\quad u = \sin t$, $du = \cos t\,dt$

$= \int_x^x u\,du$

$= \frac{u^2}{2}\Big|_x^x = \frac{1}{2}\sin^2 t\Big|_0^{2\pi}$

$= \frac{1}{2}\left[(\sin 2\pi)^2 - (\sin 0)^2\right]$

$= \frac{1}{2}(0-0)$

$= \boxed{0}$

$= \frac{1}{2}(0-0)$

$= \boxed{0}$

distance between 2 elements of an inner product space is norm of difference : $dist(f,g) = \|f-g\|$

ex. given $f(t) = t$, $g(t) = 1$ in $C[0,1]$, find distance between f and g

sol'n: $dist(f,g) = \|f-g\|$

$= \|t-1\|$

$= \sqrt{\int_0^1 (f-g)(f-g)\,dt}$

$= \sqrt{\int_0^1 (f-g)^2\,dt}$

$= \sqrt{\int_0^1 (t-1)^2\,dt}$ $\quad u = t-1$, $du = dt$

$= \sqrt{\frac{1}{3}(t-1)^3 \Big|_0^1}$

$= \sqrt{\frac{1}{3}\left[0-(-1)^3\right]}$

$= \sqrt{\frac{1}{3}(1)} = \boxed{\frac{1}{\sqrt{3}}}$

end of Exam 2 material

expect review material to be posted EOD Sat

topics from HW 11 will be on Exam 2